Maths appendix

The accumulation of a payment of 1 at the *end* of each period for n periods at a periodic interest rate i is equal to:

$$1 + (1+i)^1 + (1+i)^2 + \cdots + (1+i)^{n-1}$$

$$= \frac{(1+i)^n - 1}{i}$$

denoted by $s_{\overline{n}|i}$

The numerical example in the article had the super account of $1,000 per month before contributions tax, accumulating at a before tax rate of 7% per annum for 18 years and 5 months, to $341,708. To demonstrate this:

Let,

n = 221 months (18 years and 5 months)

$i = \frac{5.95\%}{12}$ (5.95% = net 7% after 15% tax)

$850 = monthly contribution after 15% tax

Then the end fund value is:

$850 x $(1+i)s_{\overline{n}|i}$ = $341,708

noting contributions are assumed to occur at the *beginning* of each month.

If the same account accumulated at a monthly amount and rate after a marginal tax rate of 30%, we would have:

n = 221 months (18 years and 5 months)

$j = \frac{4.90\%}{12}$ (4.90% = net 7% after 30% tax)

$700 = monthly contribution after 30% tax

Then the end fund value is:

$700 x $(1+j)s_{\overline{n}|j}$ = $251,485

And the difference between the two fund values = $90,223. This represents the accumulated difference between the two fund values under the TEIS tax benchmark. That is, an 'accumulated TEIS-style' tax concession.

The total accumulated tax concession can then be split into accumulated contributions and earnings tax concessions.

The contributions tax concession is an accumulation of $150 per month at the after marginal tax earnings rate:

= ($850 - $700) x $(1 + j)s_{\overline{n}|j}$

= $\frac{\$150}{\$700}$ x $700 x $(1 + j)s_{\overline{n}|j}$

= $\frac{\$150}{\$700}$ x $251,485

= $53,890

The accumulated earnings tax concession is the difference between the accumulation of the same $850 monthly contributions at the after super tax earnings rate and the after marginal tax earnings rate:

= $850 x ($(1 + i)s_{\overline{n}|i}$ - $(1 + j)s_{\overline{n}|j}$)

= $850 x $(1 + i)s_{\overline{n}|i}$ - $\frac{\$850}{\$700}$ x $700 x $(1 + j)s_{\overline{n}|j}$

= $341,708 - $\frac{\$850}{\$700}$ x $251,485

= $36,333

Check: $53,890 + $36,333 = $90,223

Note with the earnings concession, I have taken the difference between two accumulated values over a period of time and called that an 'accumulated TEIS-style' tax concession. However, the TEIS figures are *annual* revenue foregone estimates. I wondered therefore, could I calculate point in time concessions as per the TEIS method, and then accumulate those at the after marginal tax earnings rate and get the same result.

In month t, the interest earned in the super account is:

$850 x $i\ s_{\overline{t}|i}$ = $850 x ($(1 + i)^t - 1$)

with the concessional amount being that, multiplied by $\frac{\left((1+i) - (1+j)\right)}{((1+i)-1)}$ to adjust for the different earnings rates.

We would then accumulate that amount from month t, until month n = 221 at the after marginal tax earnings rate.

That is, the accumulated value of that concession would be:

$850 x $\frac{\left((1+i) - (1+j)\right)}{((1+i)-1)}$ ($(1 + i)^t - 1$) x $(1 + j)^{(n-t)}$

And then summing those amounts for t = 1 to n (= 221) indeed returned $36,333.

I was therefore satisfied that the difference between accumulation function values at different interest rates, is mathematically identical to the sum of the individual monthly concessions accumulated at the alternative after tax earnings rate. And therefore that presenting accumulated concessions in my article reflected the thrust of the TEIS.

That is,

\$850 x ($(1+i)s_{\overline{n}|i}$ - $(1+j)s_{\overline{n}|j}$)

$$= \$850 \text{ x } \frac{\big((1+i) - (1+j)\big)}{((1+i)-1)} \sum_{t=1}^{n} \big((1+i)^t - 1 \big) \text{ x } (1+j)^{(n-t)}$$

Note, this relationship can be proved mathematically for all n. But for ease of presentation, assume n = 5 say.

When n = 5, we want to prove that:

$$\frac{\big((1+i) - (1+j)\big)}{((1+i)-1)} \sum_{t=1}^{5} \big((1+i)^t - 1 \big) \text{ x } (1+j)^{(5-t)}$$

= $(1+i)s_{\overline{5}|i}$ - $(1+j)s_{\overline{5}|j}$

Now, let a = $(1+i)$, b = $(1+j)$. We therefore have:

$$\frac{(a-b)}{(a-1)} \sum_{t=1}^{5} \big(a^t - 1 \big) \text{ x } b^{(5-t)}$$

= $\frac{(a-b)}{(a-1)}$ $\big[(a - 1)\, b^4 + (a^2 - 1)\, b^3 + (a^3 - 1)\, b^2 + (a^4 - 1)\, b + (a^5 - 1)\big]$

noting,

$$(a^n - b^n) = (a-b) \sum_{k=0}^{n-1} a^{n-1-k}\, b^k$$

= $\frac{(a-b)}{(a-1)}$ $(a - 1)\big[b^4 + b^3(a + 1) + b^2(a^2 + a + 1) + b(a^3 + a^2 + a + 1) + (a^4 + a^3 + a^2 + a + 1)\big]$

= $(a - b)\big[(a^4 + a^3b + a^2b^2 + ab^3 + b^4) + (a^3 + a^2b + ab^2 + b^3) + (a^2 + ab + b^2) + (a + b) + 1\big]$

= $(a^5 - b^5) + (a^4 - b^4) + (a^3 - b^3) + (a^2 - b^2) + (a - b)$

= $(a + a^2 + a^3 + a^4 + a^5) - (b + b^2 + b^3 + b^4 + b^5)$

= $((1+i) + (1+i)^2 + (1+i)^3 + (1+i)^4 + (1+i)^5) - ((1+j) + (1+j)^2 + (1+j)^3 + (1+j)^4 + (1+j)^5)$

= $(1+i)s_{\overline{5}|i}$ - $(1+j)s_{\overline{5}|j}$

End of proof